for the cone at large x/d. The results shown in Figs. 1 and 2 illustrate both of these conclusions.

References

[1] Booker, H. G. and Gordon, W. E., "A theory of radio scattering in the troposhere," Proc. Inst. Radio Eng. **38,** 401 (1950).

[2] Tatarski, V. I., *Wave Propagation in a Turbulent Medium* (McGraw-Hill Book Co., New York, 1951).

[3] Lin, S. C., "Radar echoes from a marred satellite during reentry," J. Geophys. Res. **67,** 3851 (1962).

[4] Salpeter, E. E. and Treiman, S. B., "Backscatter of EM radiation from a turbulent plasma," J. Geophys. Res. **69,** 869 (1964).

[5] Batchelor, G. K., *The Theory of Homogeneous Turbulence* (Cambridge University Press, Cambridge, England, 1956) Chap. 6.

[6] Hinze, J. O., *Turbulence* (McGraw-Hill Book Co., Inc., New York, 1959) Chap. 6.

[7] Liepman, N. H. W., "Aspects of the turbulence problem," Z. Angew. Math. Phys. **III,** 321 (1952).

[8] Webb, W. H., "A model for the calculation of radar backscatter from underdense hypersonic turbulent wakes," Space Technology Lab. Rept. 6433-6005-KU-000 (June 8, 1964).

[9] Townsend, A. A., *The Structure of Turbulent Shear Flow* (Cambridge University Press Cambridge, England, 1956), Chap. 7.

[10] Lees, L. and Hromas, L., "Turbulent diffusion in the wake of a blunt-nosed body at hypersonic speeds," J. Aerospace Sci. **29,** 976 (1962).

[11] Proudian, A. P. and Feldman, S., "Some theoretical predictions of mass and electron density oscillations in wakes and comparison with experiment," AIAA Preprint 64-21 (1964).

[12] Lin, S. C. and Hayes, J. E., "A quasi one-dimensional model for chemically reacting turbulent wake of hypersonic objects," AIAA Preprint 63-449 (1963).

[13] Webb, W. H. and Hromas, L. A., "Turbulent diffusion of a reacting wake," AIAA Preprint 64-42 (1964).

A Transient Solution of the Fokker-Planck Equation

C. F. EATON*

Raytheon Company, Bedford, Mass.

Introduction

In many cases of practical interest, it is found that the mean temperature of the electrons in an ionized gas is different from that of the gas molecules. Such cases exist, for example, when an initially hot electron gas is injected into a relatively cold neutral gas, or when the neutral molecule temperature and density are changing. The electrons do not follow these changes immediately, as many collisions of an electron with the neutral molecules are required before the electrons come into thermal equilibrium. In what follows, it is shown that a complete solution of the distribution function is forthcoming when the collision frequency is velocity independent, corresponding to the Maxwell law of interaction.

Fokker-Planck Equation

The Boltzmann equation is usually written as

$$(\partial f/\partial t) + \nabla f\cdot \mathbf{v} + \nabla_v f\cdot \mathbf{g} = C \tag{1}$$

where

f = velocity distribution function
t = time
C = collision integral

and $\nabla f\cdot\mathbf{v}$ and $\nabla_v f\cdot \mathbf{g}$ represent the influences of diffusion and external forces, respectively.

The collision integral is given by Allis[1] as

$$C = \frac{m}{Mv^2}\frac{d}{dv}\left[v^2\nu\left(vf + \frac{kT}{m}\frac{df}{dv}\right)\right] \tag{2}$$

where

m = electron mass
M = molecular mass
v = electron velocity
ν = collision frequency
k = Boltzmann's constant
T = gas temperature

Equation (2) is actually the first term of the isotropic part of the collision integral for elastic collisions of electrons and molecules expanded in powers of m/M and is therefore a good approximation for the case to be considered here.

For a spatially uniform gas in which no external forces are acting, the Boltzmann equation is then written as

$$\frac{\partial f}{\partial t} = \frac{m}{Mv^2}\frac{\partial}{\partial v}\left[v^2\nu\left(vf + \frac{kT}{m}\frac{\partial f}{\partial v}\right)\right] \tag{3}$$

which is the usual form of the Fokker-Planck equation.

General Solution of the Fokker-Planck Equation

The collision frequency is, in general, a function of velocity, and it is expressed here as a general power series expansion in v, i.e.,

$$\nu = \Sigma \nu_n v^n \tag{4}$$

where the values of n may be positive and negative.

The Maxwellian distribution is written as

$$f = (A/\pi)^{3/2}e^{-Av^2} \tag{5}$$

where A is a function of time only. Substituting Eqs. (4) and (5) into Eq. (3) gives the following differential equation for the function A:

$$\frac{M}{m}\left(\frac{3}{2} - v^2A\right)\frac{dA}{dt} = A\Sigma\nu_n v^n \times \left\{n + 3 - 2A\left[\frac{kT}{m}(n+3) + v^2\right] + \frac{4kT}{m}A^2v^2\right\} \tag{6}$$

An examination of Eq. (6) reveals that, since A is independent of the velocity, the *only* nontrivial, consistent equation results when the collision frequency is velocity independent, i.e., when $\nu = \nu_0$, corresponding to the Maxwell law of interaction. The resulting equation for this case is

$$\frac{M}{m\nu}\frac{dA}{dt} + \frac{4kT}{m}A^2 - 2A = 0 \tag{7}$$

In general, both ν and T have general time variations. Consider the homogeneous equation

$$(M/m)\,(dA/d\tau) = 2A \tag{8}$$

where $d\tau = \nu dt$. The solution of Eq. (8) is just

$$A = e^{(2m/M)\tau} \tag{9}$$

The solution of the nonhomogeneous equation (7) is assumed to be of the form

$$A = Be^{(2m/M)\tau} \tag{10}$$

Substitution of Eq. (10) into Eq. (7) results in the following equation for B:

$$\frac{M}{m\nu}\frac{dB}{dt} = -\frac{4kT}{m}B^2e^{(2m/M)\tau} \tag{11}$$

Presented as Preprint 64-382 at the AIAA 1st Annual Meeting, Washington, D. C., June 29–July 2, 1964.

* Staff Engineer.

Table 1 Various solutions of Eq. (14) for specific cases of temperature variations ($d\nu/dt = 0$)

T	T_e
$dT/dt = 0$	$T(1 - e^{-\nu\beta t}) + T_0 e^{-\nu\beta t}$ (also developed in Ref. 3)
$T_{0g}(1 + at)$	$T_{0g}(1 + at - a/\nu\beta) + e^{-\nu\beta t} \times [T_0 - T_{0g}(1 - a/\nu\beta)]$
$T_{0g} + T_a e^{at}$	$T_{0g} + T_a[\nu\beta/(a + \nu\beta)]e^{at} + \{T_0 - T_{0g} - T_a[\nu\beta/(a + \nu\beta)]\}e^{-\nu\beta t}$
$T_{0g} + T_a \sin at$	$T_{0g} + T_a[\beta\nu/(a^2 + \beta^2\nu^2)](\beta\nu \sin at - a \cos at) + \{T_0 - T_{0g} + T_a[a\beta\nu/(a^2 + \beta^2\nu^2)]\}e^{-\beta\nu t}$

$$B^{-1} = \int \frac{4kT\nu}{M} \left\{\exp\left[\left(\frac{2m}{M}\right) \int \nu dt'\right]\right\} dt \tag{12}$$

and the general solution of A is then written as

$$A = \frac{\exp[(2m/M)\int \nu dt]}{\int (4\,kT\nu/M)[\exp\{(2m/M)\int \nu dt'\}]\,dt} \tag{13}$$

Since the electron temperature T_e is just $m/2kA$, Eq.(13) gives

$$T_e = \exp(-\beta \int \nu dt)\ \{\int \beta\, T\nu \times [\exp(\beta \int \nu dt')]\,dt + K\} \tag{14}$$

where $\beta = 2m/M$ and K is the integration constant.

Closed Form Solutions for T_e

In Table 1 are given various solutions of Eq. (14) for some specific cases of temperature variations. The initial electron temperature in Table 1 is T_0.

Approximate Solution of the General Problem

It would be desirable to have a solution to the general case in which both the gas temperature and density can have quite arbitrary temporal variations. Since any reasonable arbitrary temperature variation may be divided into intervals, and the ith interval approximated by $T_i = T_{0gi}\,(1 + a_i t)$, and similarly for the collision frequency, it is worthwhile to find a solution for this case which may be used successively on each interval to give an approximate solution to the over-all problem.

Integrating by parts, the integral in the braces of Eq. (14) becomes

$$B\int T\nu \exp(\beta \int \nu dt')\,dt = T \exp(\beta \int \nu dt) - \int \exp(\beta \int \nu dt')\,(dT/dt)\,dt \tag{15}$$

Making the substitutions $T = T_{0g}\,(1 + at)$, $\nu = \nu_0\,(1 + bt)$, Eq. (15) becomes

$$\beta \int T\nu(\beta \int \nu dt')dt = T_{0g}\,(1 + at) \times \exp[\beta\nu_0 t\,(1 + bt/2)] - aT_{0g} \int \exp[\beta\nu_0 t\,(1 + bt/2)] \times dt = T_{0g}\,(1 + at) \exp[\beta\nu_0 t\,(1 + bt/2)] + [aT_{0g}/(\beta b\nu_0)^{1/2}] \exp[-\beta\nu_0/2b]\ \mathrm{erf}\{(2\beta\nu_0)^{1/2} \times [t(b/2)^{1/2} + 1/(2b)^{1/2}]\} + C \tag{16}$$

where erf (x) denotes the error function.[2] Combining Eqs. (16) and (14) gives the final solution as

$$T_e = T_{0g}\,(1 + at) + [aT_{0g}/(\beta\nu_0 b)^{1/2}] \times \exp[-\beta\nu_0\,(t + bt^2/2 + 1/2b)]\ \mathrm{erf}\{(2\beta\nu_0)^{1/2} \times [t(b/2)^{1/2} + 1/(2b)^{1/2}]\} + \{T_0 - T_{0g} \times (1 + [a/(\beta\nu_0 b)^{1/2}] \exp(-\beta\nu_0/2b)\ \mathrm{erf}(\beta\nu_0/2)^{1/2})\} \times \exp[-\beta\nu_0 t\,(1 + bt/2)]$$

References

[1] Allis, W. P., "Motions of ions and electronics," *Handbuch der Physik* (Springer-Verlag, Berlin, 1955), Vol. XXI, p. 411.

[2] Fried, B. D. and Conte, S. D., *The Plasma Dispersion Function* (Academic Press, New York, 1961), tabulation of the function erf(X) in combined form for complex values of x; see also Bhatnagar, P. L., Gross, E. P., and Krook, M., "A model for collision processes in gases I. Small amplitude processes in charged and neutral one-component systems," Phys. Rev. **94**, 511–525 (1963); also Rosser, J. B., *Theory and Application of* $\int_0^z e^{-x^2}$ (Mapleton House, Brooklyn, N. Y., 1948); also Salzer, H. E., "Formula for calculating the error function of a complex variable," Math. Tables Aids Computations **35**, 67 (1951).

[3] Osipov, D. I., "Conservation of the form of the Maxwellian distribution in a relaxing gas," AIAA J. **1**, 261–262 (1963).

Aerodynamic Pitching Derivatives of a Wedge in Hypersonic Flow

J. P. Appleton*

University of Southampton, Southampton, England

Introduction

In a recent series of experiments carried out in the University of Southampton gun tunnel, East[1] has measured the aerodynamic stiffness and damping derivatives m_θ and $m_{\dot\theta}$, respectively [see Eqs. (20–22) for definitions], at a freestream Mach number $M_\infty \doteqdot 9.7$ of a two-dimensional wedge aerofoil oscillating in pitch about several positions along its chord; the experimental results for the sharp leading edge model are illustrated in Fig. 1. Since the wedge half angle δ was equal to $9\frac{1}{2}°$ giving a thickness parameter $M_\infty\delta = 1.6$, he compared his measured values of m_θ and $m_{\dot\theta}$ with theoretical predictions that embodied the "strong-shock" piston theory suggested by Miles.[2] The agreement obtained between the experimental results and the theory was certainly more than qualitative. Within the limits of experimental scatter, the measured values of m_θ were well correlated by the theory, although the experimental values of $m_{\dot\theta}$ showed a certain skewness about the theoretical prediction which made them fall below the theoretical prediction for forward positions of the pivot point and above the theoretical prediction for aft positions of the pivot point. Indeed, the experimental results indicated positive values of the damping derivative for positions of the pivot between 0.2 and 0.4 chord. East attributed the discrepancies between experiment and theory primarily to the interference of the flow along the two sides of the two-dimensional model which was caused by laminar

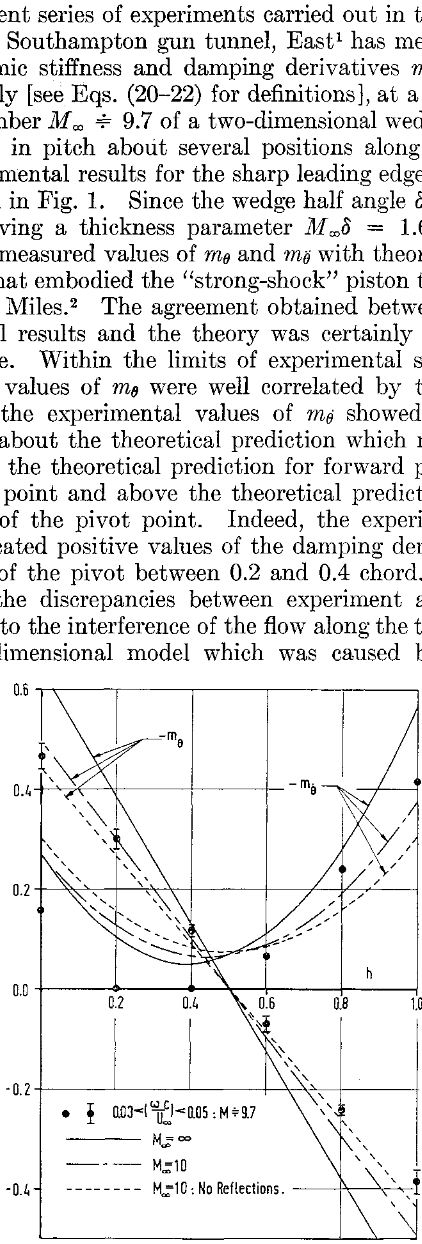

Fig. 1 Variation of the aerodynamic stiffness and damping derivatives with pivot position.

Received June 22, 1964.

* Lecturer, Department of Aeronautics and Astronautics; now Senior Reseach Engineer, General Motors Defense Research Laboratories, Santa Barbara, Calif. Member AIAA.